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DETERMINATION OF THE SECOND HOMOLOGY AND COHOMOLOGY GROUPS OF A SPACE BY MEANS OF HOMOTOPY INVARIANTS

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1. *Introduction.*—The purpose of this note is to give a description of the 2-dimensional homology and cohomology groups of a topological[1] space in terms of homotopy invariants. Besides using the first and the second homotopy groups the description uses a certain new 3-dimensional invariant which will be described below.

The method utilizes the cohomology theory of abstract groups recently developed by the authors.[2] Given a group π and an additive abelian group G with π as (left) operators, we define the group $C^q(\pi, G)$ of q-dimensional cochains as the group of all functions of q variables in π with values in G. The coboundary δf of a q-cochain f is defined by

$$\begin{aligned}(\delta f)(x_1, \ldots, x_{q+1}) &= x_1 f(x_1, \ldots, x_{q+1}) \\ &+ \sum_{i=1}^{q} (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{q+1}) \\ &+ (-1)^{q+1} f(x_1, \ldots, x_q).\end{aligned}$$

Cochains f with $\delta f = 0$ are called *cocycles;* they form a subgroup $Z^q(\pi, G)$ of $C^q(\pi, G)$. Cochains of the form δf, where $f \in C^{q-1}(\pi, G)$ are called coboundaries, form a subgroup $B^q(\pi, G)$ of $C^q(\pi, G)$. Since $\delta\delta f = 0$, B^q is a subgroup of Z^q. The quotient group Z^q/B^q is called the qth cohomology group of π over G and is denoted by $H^q(\pi, G)$.

2. *The Invariant k^3.* Let X be an arcwise connected topological space with base point x_0. Consider the fundamental group $\pi_1 = \pi_1(X, x_0)$. For every $w \in \pi_1$ select a path (i.e., a singular 1-simplex) $R(w)$ representing the element w. Given two elements $w_1, w_2 \in \pi_1$, and given an ordered 2-simplex s_2 with vertices v_0, v_1, v_2, construct a singular 2-simplex $R(w_1, w_2): s_2 \to X$

such that the edges v_0v_1, v_1v_2 and v_0v_2 are mapped according to $R(w_1)$, $R(w_2)$ and $R(w_1w_2)$. Now let $w_1, w_2, w_3 \in \pi_1$, and let s_3 be a 3-simplex with ordered vertices v_0, v_1, v_2, v_3. Consider the faces $v_1v_2v_3$, $v_0v_2v_3$, $v_0v_1v_3$ and $v_0v_1v_2$ of s_3 and map them into X according to the maps $R(w_2, w_3)$, $R(w_1w_2, w_3)$, $R(w_1, w_2w_3)$ and $R(w_1, w_2)$. There results a map $R(w_1, w_2, w_3)$ of the boundary $B(s_3)$ of s_3 into X. Regarding v_0 as the base point of $B(s_3)$, this determines uniquely an element $k^3(w_1, w_2, w_3)$ of the second homotopy group[3] $\pi_2 = \pi_2(X, x_0)$. We shall regard k^3 as an element of the group $C^3(\pi_1, \pi_2)$ of the 3-dimensional cochains of π_1 with coefficients in π_2, with π_1 operating on π_2 in the usual way.

THEOREM 1. *k^3 is a cocycle. A change of the representatives $R(w_1)$ and $R(w_1,w_2)$ alters k^3 by coboundaries. Thus k^3 determines uniquely a cohomology class $\mathbf{k}^3 \in H^3(\pi_1, \pi_2)$ which is a topological invariant of (X, x_0). Moreover, given any cocycle $k^3 \in Z^3(\pi_1, \pi_2)$ belonging to the class $\mathbf{k}^3$, there is a choice of representatives $R(w)$, $R(w_1, w_2)$ which produce k^3.*

To see that k^3 is a cocycle, let $w_1, w_2, w_3, w_4 \in \pi_1$, and consider a 4-simplex s_4 with ordered vertices v_0, v_1, v_2, v_3, v_4. It is possible to map the boundaries of faces $v_1v_2v_3v_4$, $v_0v_2v_3v_4$, $v_0v_1v_3v_4$, $v_0v_1v_2v_4$, $v_0v_1v_2v_3$ by the maps $R(w_2, w_3, w_4)$, $R(w_1w_2, w_3, w_4)$, $R(w_1, w_2w_3, w_4)$ and $R(w_1, w_2, w_3)$. There results a map into X of the 2-dimensional skeleton of s_4. From the definition of addition in π_2 and of the operators it can be shown that

$$w_1k^3(w_2, w_3, w_4) - k^3(w_1w_2, w_3, w_4) + k^3(w_1, w_2w_3, w_4) - k^3(w_1, w_2, w_3w_4) + k^3(w_1, w_2, w_3) = 0;$$

i.e., $\delta k^3 = 0$.

Suppose that, while keeping the representatives $R(w)$ unchanged, we replace $R(w_1, w_2)$ by new representatives $R'(w_1, w_2)$. The two maps of the 2-simplex s_2 determine an element $h(w_1, w_2)$ of π_2, and again from the definition of π_2 it follows that

$$k^3(w_1, w_2, w_3) - k'^3(w_1, w_2, w_3) = w_1h(w_2, w_3) - h(w_1w_2, w_3) + h(w_1, w_2w_3) - h(w_1, w_2);$$

i.e., that $k^3 - k'^3 = \delta h$. Since h can be chosen arbitrarily, k^3 may be altered by an arbitrary coboundary.

Finally, if the representatives $R(w)$ are replaced by $R'(w)$, then, since $R(w)$ and $R'(w)$ are homotopic (with end-points fixed), new representatives $R'(w_1, w_2)$ may be chosen in such a way that $k'^3 = k^3$.

3. The Second Homology Groups of X.—In terms of π_1, π_2 and the new invariant $\mathbf{k}^3 \in H^3(\pi_1, \pi_2)$, we can describe the 2-dimensional homology and cohomology groups of X.

Let $k^3 \in Z^3(\pi_1, \pi_2)$ be a cocycle in the class $\mathbf{k}^3$, and let G be a (topological)

abelian group selected as coefficient group for cohomologies. Consider the additive group of pairs (g_1, g_2) such that

(3.1) $g_1 \in C^2(\pi_1, G)$; i.e., g_1 is a function of two variables in π_1 with values in G,

(3.2) $g_2 \in \mathrm{Hom}(\pi_2, G)$; i.e., g_2 is a homomorphism of π_2 into G,

(3.3) $(\delta g_1)(w_1, w_2, w_3) = g_2(k^3(w_1, w_2, w_3))$,

(3.4) $g_2(w\alpha) = g_2(\alpha)$ for $w \in \pi_1$, $\alpha \in \pi_2$.

Let $H^2(\pi_1, \pi_2, k^3, G)$ be the quotient group of all these pairs (g_1, g_2) by the subgroup of pairs of the form $(\delta h, 0)$ where $h \in C^1(\pi_1, G)$.

THEOREM 2. *The second cohomology group $H^2(X, G)$ of X over G is isomorphic with the group $H^2(\pi_1, \pi_2, k^3, G)$.*

Let $f \in Z^2(X, G)$ be a singular cocycle. f induces a homomorphism of the integral homology group $H_2(X) \to G$; combined with the natural homomorphism $\pi_2 \to H_2(X)$ this produces a homomorphism $g_2(f)$: $\pi_2 \to G$ which satisfies (3.4).

Given the cocycle $k^3 \in Z^3(\pi_1, \pi_2)$, it follows from Theorem 1 that the representatives $R(w)$ and $R(w_1, w_2)$ may be selected to produce k^3. The value of the cocycle f on the singular 2-simplex $R(w_1, w_2)$ gives an element $g_1(f)$ of $C^2(\pi_1, G)$. The pair $(g_1(f), g_2(f))$ satisfies conditions (3.1)–(3.4). If $f = \delta f'$ for a singular 1-cochain f', then $(g_1(f), g_2(f)) = (\delta h, 0)$ where $h(w)$ is the value of f' on the singular 1-simplex $R(w)$. The correspondence $f \to (g_1(f), g_2(f))$ thus yields a homomorphism $H^2(X, G) \to H^2(\pi_1, \pi_2, k^3, G)$. The proof that this is an isomorphism onto is straightforward.

The results for the homology groups follow by the application of the Pontrjagin character theory.

4. Decomposition of the Second Cohomology Groups.—Let π_2° denote the subgroup of π_2 spanned by the elements $\alpha - w\alpha$, $\alpha \in \pi_2$, $w \in \pi_1$.

The correspondence $(g_1, g_2) \to g_2$ maps $H^2(\pi_1, \pi_2, k^3, G)$ homomorphically onto a subgroup J of $\mathrm{Hom}(\pi_2, G)$. The kernel of this homomorphism is the cohomology group $H^2(\pi_1, G)$. In the cohomology group $H^2(X, G)$ this kernel corresponds to the subgroup $\Lambda^2(X, G)$ consisting of those cohomology classes which annihilate all integral spherical cycles. The isomorphism $H^2(\pi_1, G) \approx \Lambda^2(X, G)$ is known.[4]

The quotient group $H^2(X, G)/\Lambda^2(X, G)$ is isomorphic with the subgroup J of $\mathrm{Hom}(\pi_2, G)$. In view of (3.4) every homomorphism in J must annihilate π_2°. A homomorphism g: $\pi_2 \to G$ that annihilates π_2° induces a homomorphic mapping g^*: $H^3(\pi_1, \pi_2) \to H^3(\pi_1, G)$. The group J consists of those homomorphisms g: $\pi_2 \to G$ which annihilate π_2° and for which $g^*k^3 = 0$.

If $k^3 = 0$, then $H^2(\pi_1, G)$ is a direct summand of $H^2(\pi_1, \pi_2, k^3, G)$, and J consists of the annihilators of π_2°. In this case we then have a direct sum decomposition

$$H^2(X, G) \approx H^2(\pi_1, G) + \text{Hom}\,(\pi_2/\pi_2^\circ, G).$$

This implies that in this case π_2° is the kernel of the natural homomorphism of π_2 into the 2-dimensional integral homology group of X. Comparing this with recent results of Hopf[5] yields numerous examples of spaces with $k^3 \neq 0$.

5. *Generalizations.*—1°: The construction of k^3 generalizes in an obvious fashion to yield a cohomology class $k^{n+1} \in H^{n+1}(\pi_1, \pi_n)$, provided that $\pi_i = 0$ for $1 < i < n$. Theorems 1 and 2 generalize accordingly.

2°: Assuming $\pi_3 = 0$, the groups $H^3(X, G)$ may be expressed by means of π_1, π_2 and k^3. The algebraic constructions involved are quite cumbersome.

3°: If π_1 acts as a group of operators on G, and $H^2(X, G)$ denotes the cohomology group of X with G as local coefficients,[6] Theorem 2 is still valid provided that (3.4) is replaced by $g_2(w\alpha) = wg_2(\alpha)$.

4°: Instead of working in the space X itself, as above, one could use the universal covering space $\tilde{X}$ of X and regard π_1 as a group of operators on $\tilde{X}$. In this approach homotopy arguments are replaced by homology arguments, and some additional generality is gained. This approach will be used in the full exposition elsewhere of the results.

[1] Singular homologies and cohomologies are used throughout; for these concepts see Eilenberg, S., *Ann. Math.*, **45**, 407–447 (1944).

[2] Eilenberg, S., and MacLane, S., *Bull. Amer. Math. Soc.*, **50**, 53 (1944); *Proc. Nat. Acad. Sci. U. S. A.*, **29**, 155–158 (1943); *Ann. Math.*, **46**, 480–509 (1945); also forthcoming papers in *Ann. Math.*

[3] See Robbins, H., *Trans. Amer. Math. Soc.*, **49**, 319 (1941).

[4] Hopf, H., *Comment. Math. Helv.*, **14**, 257–309 (1942); *Ibid.*, **15**, 27–32 (1942); *Ibid.*, **17**, 39–79 (1944); see also papers in reference 2; Freudenthal, H., *Ann. Math.*, **47**, 274–316 (1946); Eckmann, B., *Comment. Math. Helv.*, **18**, 232–282 (1946).

[5] Hopf, H., *Comment. Math. Helv.*, **17**, 307–326 (1945).

[6] Steenrod, N. E., *Ann. Math.*, **44**, 610–627 (1943); Eilenberg, S., Homology Theory of Spaces with Operators I, *Trans. Amer. Math. Soc.* (in print).

RATIONAL FUNCTIONS OF A COMPLEX VARIABLE AND RELATED POTENTIAL CURVES

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1. We shall study curves related to rational fractional functions of a complex variable $z = x + iy$. Our new theorems are extensions of known results concerning curves related to rational integral functions of z.

2. Curves defined by setting the real part of a polynomial (rational integral function) in z equal to zero, are well known. These were studied initially by Briot and Bouquet, and Bocher; and later were characterized completely by Kasner. These curves were called *algebraic potential curves* by Kasner in 1901. This term is used in later papers by Loria and in the German Encyclopedia. In the present article, we shall find it convenient to use the terminology: *polynomial potential or harmonic curves.*

We shall say that an algebraic curve is a *rational potential or rational harmonic curve* if it is obtained by setting equal to zero the real part of a rational fractional function of z. The class of rational harmonic curves of course includes the class of polynomial harmonic curves.

By the degree r of a rational function is meant the maximum of the two degrees of the numerator and denominator.

If the degree of the rational function of z is r, the rational harmonic curve is given by the equation: $P(x, y) = 0$, where P is a polynomial in (x, y) of degree $(2r - k)$ where $0 \leqq k \leqq r$.

In general, $P(x, y)$ does not satisfy the Laplace equation, but it does satisfy a certain partial differential equations of fourth order.

3. *A rational harmonic curve of degree $(2r - k)$ has k real asymptotes. If $2 \leqq k \leqq r$, the k real asymptotes are concurrent and make equal angles with one another. The angle between consecutive asymptotes is π/k. The remaining $2(r - k)$ asymptotes are minimal. This last result remains valid even when $k = 0$ or 1; and also the first result remains true trivially.*

When $k = r$, this new theorem reduces to the theorem of Briot and Bouquet concerning the asymptotes of a polynomial harmonic curve.[1] In this case, there are no minimal asymptotes.

4. *An algebraic curve is rational harmonic of degree not exceeding $(r + s)$ where $r > 1$ and $s > 1$, if and only if it passes through the r^2 foci of a curve C_r of class r and the s^2 foci of a curve C_s of class s such that no minimal line contains a focus of C_r and a focus of C_s.*

Two analytic curves are said to be *conjugate* if they are obtained by setting the real and imaginary parts of an analytic function of z equal to zero.

A pair of algebraic curves of degree not exceeding $(r + s)$, are conjugate rational harmonic if and only if they intersect orthogonally in the r^2 foci of a set of confocal curves C_r and in the s^2 foci of another set of confocal curves C_s such that no minimal line contains a focus of a C_r and a focus of a C_s.

When s or r is zero, these characterizations of rational harmonic curves by focal properties reduce to the corresponding theorems obtained by Kasner for polynomial potential curves.[2]

5. Characterizations of polynomial harmonic curves have been given by Kasner by use of the concepts of apolarity and polarity. We shall discuss the apolar and polar properties of rational harmonic curves in later

work. It is remarked that although the polars of any polynomial harmonic curve are also polynomial harmonic; the corresponding result for rational potential curves is not true; that is, the polar of a rational harmonic curve is not rational harmonic in general.

6. *The degree of the Schwarzian reflection or conformal symmetry with respect to a rational harmonic curve of degree not exceeding* $(r + s)$, *where* $0 \leqq s \leqq r$, *is exactly* r^2. *The satellite of a rational harmonic curve is the same curve.*

When $s = 0$, this result reduces to the corresponding theorem of Kasner with respect to the polynomial potential curves.[3]

7. We shall say that an algebraic curve is *inverse rational* (*or inverse polynomial*) *harmonic* if it is obtained by setting equal to zero, the real part of the inverse of a rational fractional (or integral) function of z.

All inverse rational harmonic curves are unicursal. Conversely all unicursal curves can be identified with inverse rational harmonic curves.

A relationship between an inverse rational harmonic curve and its associated rational function of z is given by the following result.

If the rational function of z *is of degree* r, *the degree of the inverse rational harmonic curve is* $(2r - k)$ *where* $0 \leqq k \leqq r$. *There are exactly* $2(r - k)$ *minimal asymptotes and at most* k *real asymptotes.*

Of course, a pair of conjugate inverse rational harmonic curves of degree $(2r - k)$ will intersect orthogonally in $(2r - k)^2$ points, at most.

All inverse polynomial harmonic curves of degree r *are the special unicursal curves tangent to the line at infinity in one real point, the order of contact being* $(r - 1)$.

8. The only conic sections which are rational harmonic are the circles and equilateral hyperbolas. The latter are the only conics which are polynomial harmonic.

All conic sections are inverse rational harmonic. Parabolas are the only conic sections which are inverse polynomial harmonic.

If a cubic curve is rational harmonic, it is either a special circular cubic or else it is a cubic curve with three real asymptotes which are concurrent and make an angle of $\pi/3$ with one another. The cubic curves of the latter type are the only polynomial harmonic cubic curves.

[1] Briot and Bouquet, *Theorie des fonctions elliptiques*, Book IV, Chapter II, p. 226. Paris-Gauthier-Villar (1875). See also Bocher, Güttingen prize memoir.

[2] Kasner, "On the Algebraic Potential Curves," *Bull. Amer. Math. Soc.*, **7**, 392–399 (1901). Also "Some Properties of Potential Surfaces," *Ibid.*, **8**, 243–248 (1902).

[3] Kasner, "Algebraic Curves, Symmetries and Satellites," these PROCEEDINGS, **31**, 250–252 (1945). Also *La satelite conforme de una curva algebraica general*, Revista de la Union Matematica Argentina, **2**, 77–83 (1946) (Buenos Aires).

INTEGRAL AND RATIONAL NUMBERS IN THE NUCLEAR DOMAIN AND THEIR SIGNIFICANCE

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Using the value of the fine structure constant given by Birge[1] in 1941

$$hc/e^2 = 2\pi/\alpha = 860.986 \pm 0.101. \quad (1)$$

This number is so very close to an integer that it is natural to assume that

$$hc/e^2 = 861 \quad (2)$$

exactly. In this paper we shall assume that equation (2) is *true* and *significant*. Note that

$$861 = {}^1/_2(42 \times 41) \quad (3)$$

so that 861 is of the form ${}^1/_2n\ (n-1)$ with $n = 42$. Both these numbers 861 and 42 turn up in very interesting ways in connection with nuclei and elementary particles, as will be shown in the sequel. Also $42 = 6 \times 7$ and the number 7 also appears frequently in connection with some of these quantities. The quantities concerned are the masses and binding energies of nuclei, both in the ground state and in excited states, the magnetic moments of nuclei and the ratios of the masses of the proton and neutron to that of the electron.

The mass of the H^1 atom on the physical scale is 1.00813 ± 0.000017 according to Birge.[1] Subtracting the mass of the electron on the physical scale, the mass of the proton is

$$M_p = 1.00758 \pm 0.000017 \text{ m. u.} \quad (4)$$

Now

$$2M_p/861 = 0.00234049 \text{ m. u.} \quad (5)$$

which is the binding energy of the deuteron within the limits of experimental error.

Using as the mass of the H^1 atom the value given above and as the mass of the neutron 1.00893, table 1 gives the binding energies of the nuclei up to He^4 on the supposition that the constituents are protons and neutrons. The values of the atomic masses used are taken from Appendix 6 in *Applied Nuclear Physics* by Pollard and Davidson.[2]

TABLE 1

ISOTOPE	ATOMIC MASS	BINDING ENERGY IN M. U.	BINDING ENERGY ÷ 0.0023405	12 × COLUMN 4	NEAREST INTEGER TO COLUMN 5
H^2	2.01472	0.00234	1.000	12.00	12
H^3	3.01704	0.00895	3.824	45.89	46
He^3	3.01701	0.00818	3.495	41.94	42
He^4	4.00388	0.03024	12.920	155.04	155

The numbers in column five of table 1 are integral within the limits of error of column three. These integers are given in column six. This table suggests that $2\,M_p/861$ and $M_p/6 \times 861$ are natural units for the binding energies of nuclei.

$$M_p/6 \times 861 = M_p/5166 = 0.000195041 \text{ m. u.} \qquad (6)$$

However, someone may suggest that since the mass of the neutron, M_n, is not very different from the mass of the proton that the quantities $2M_n/861$ and $M_n/5166$ would serve just as well or better than the quantities above. Actual trial shows that in that case the numbers in column five would not approximate integers as closely as those in table 1.

For convenience in further discussion names must be chosen for these units. $2M_p/861$ will be designated a *deuterium unit*, which will often be abbreviated to D. U. in writing. This shall be regarded as a unit of mass or energy as circumstances require.

$$^1/_{12} \text{ D. U.} = M_p/5166 \qquad (7)$$

shall be designated a *prout*, the abbreviation for which shall be Pr. First note that

$$M_p = 5166 \text{ Pr.} = 7 \times 738 \text{ Pr.} \qquad (8)$$

Now consider the mass of the neutron.

$$\left.\begin{aligned} M_n - M_p &= 0.00135 \text{ m. u.} \\ &= 6.92 \text{ Pr.} \end{aligned}\right\} \qquad (8a)$$

Within the limits of experimental error $(M_n - M_p)$ is 7 seven prouts and we shall assume that it is 7 *exactly* so that

$$M_n = 5173 \text{ Pr.} = 7 \times 739 \text{ Pr.} \qquad (9)$$

exactly. Note the two occurrences of the number 7 in equations (8) and (9).

These results combined with the results in table 1 lead to the conclusion that the *masses of all the nuclei in table 1 in the ground state are an integral number of prouts.*

This suggests the working hypothesis that the *masses of all stable nuclei in the ground state are an integral number of prouts* (H^3 is of course not stable, but it fits into the integral rule nevertheless, judging by table 1).

It is now very easy to calculate the nuclear mass in prouts of all the isotopes in table 1.

The sum of the nuclear masses of H^1, n, H^2, H^3, He^3 and He^4 is 72,118 prouts exactly. The sum of these nuclear masses is 14.06586 m. u. Hence by division

$$1 \text{ Pr.} = 0.0001950396 \text{ m. u.} \tag{10}$$

And

$$M_p = 1.007574 \text{ m. u.} \tag{11}$$

$$M_n = 1.008940 \text{ m. u.} \tag{12}$$

These values should be good to seven significant figures.

A good nucleus on which to test our idea that all masses of stable nuclei are an integral number of prouts is that of O^{16}, since by definition its mass is exactly 16.000000 m. u. Using (11) and (12) its binding energy is 0.136504 m. u., which by equation (10) gives 699.878 prouts. There is no difficulty in identifying this as 700 prouts exactly.

With this information it is possible to calculate astoundingly accurate values of 1 prout, 1 D. U., M_p and M_n. Using Birge's value[1] of the mass of the electron on the physical scale and estimating the binding energy of the electrons in O^{16} as 0.0000020 m. u., the nuclear mass of O^{16} is 15.9956130 $\pm$ 0.0000021 m. u. (13)

The nuclear mass of 0^{16} is 82,012 prouts. Hence by division

$$1 \text{ prout} = 0.000195039909 \pm 26 \times 10^{-12} \text{ m. u.} \tag{14}$$

$$1 \text{ D. U.} = 0.00234047891 \pm 31 \times 10^{-11} \text{ m. u.} \tag{15}$$

$$M_p = 1.00757617 \pm 13 \times 10^{-8} \text{ m. u.} \tag{16}$$

$$M_n = 1.00894145 \pm 13 \times 10^{-8} \text{ m. u.} \tag{17}$$

$$\text{Mass of } H^1 = 1.00812478 \pm 13 \times 10^{-8} \text{ m. u.} \tag{18}$$

If one uses these very accurate values to calculate the binding energies of nuclei up to Ne^{21} from the experimentally determined atomic masses and then reduces these binding energies to prouts, one finds that in all except a few cases the results are remarkably close to an integral number of prouts. We will cite only two cases. Ewald[3] has determined the mass of C^{13} as 13.007581 $\pm$ 0.000025. This yields a binding energy of 531.974 prouts, which is easily identified as 532 prouts. His value for N^{15} is 15.004934 $\pm$ 0.000030, which yields a binding energy of 633.046 prouts, which is very close to 633 prouts. The evidence is very strong in favor of our hypothesis, but it is too extensive to be given here. Above Ne^{21} the present atomic mass values appear to be less accurate, but it is natural

to assume that our hypothesis is valid throughout the entire range of nuclear masses.

It is interesting to note that out of a total of 18 nuclear masses which were determined in integral prouts, 6 of the values were divisible by 7. It seems to be true also that *the nuclear masses in prouts of all the most abundant isotopes are divisible by 7*, although the converse of this is not true. Using the values in equations (14), (17) and (18), the binding energy in prouts per particle was calculated for a very large number of nuclei scattered throughout the periodic table. Below Ne^{21} these values can be obtained exactly, above Ne^{21} in most cases only approximately. It seems to be true that this quantity can never exceed the integral value of 48 prouts (4 D. U.) per particle, although it comes very close to 48 in a number of cases. In fact this quantity probably lies between 47 and 48 for all stable nuclei between $Z = 22$ and $Z = 42$. Between $Z = 1$ and $Z = 21$ it rises from 6 to a value between 47 to 48. Between $Z = 42$ and $Z = 84$ it drops gradually to about 42 prouts per particle. Above $Z = 84$ it may fall below 42 prouts per particle.

The results just given together with other evidence lend some support to the idea that each "shell" of protons in the nucleus has 42 protons. For it is at $Z = 43$ that the binding energy per particle begins to fall below the plateau value between 47 and 48.

Summing up the results so far, there is very considerable evidence for the following ideas:

1. There is a natural unit of mass and energy for the proton, neutron and all other stable nuclei, which we have designated the prout. In terms of this unit the masses of the neutron and all stable nuclei in the ground state are integral numbers.

2. The binding energies of stable nuclei in the ground state are an integral number of prouts.

3. The integers 42 and also 7, 861, 5166 and 5173 play an important rôle in "explaining" the masses of the proton, neutron and all nuclei. As mentioned before the number 42 plays an important rôle in the behavior of protons when combined into nuelei. One may therefore say that 42 and also probably 7 plays an important rôle in the basic structure of the universe.

It is perfectly clear that the masses of nuclei in excited states cannot always be an integral number of prouts, although this might be and evidently is true for some excited states. For the experiments with slow neutrons demonstrate the existence of energy levels with a spacing as small as 10 electron volts. This immediately suggests a natural generalization of the working hypotheses considered above. It is:

4. The binding energies of unstable nuclei and of nuclei in excited states are always a rational number of prouts.

5. The masses of unstable nuclei and of nuclei in excited states are always a rational number of prouts.

If these assumptions are correct then the disintegration energies in α-ray emission and the energies of γ-rays must be a rational number of prouts. These conclusions have been tested on some of the best data for α-rays and γ-rays and conclusions 4 and 5 are corroborated in sufficient number of cases to lend plausibility to these conclusions. Naturally the test can be satisfactory only when the denominator of the rational number mentioned is not too large when the rational number is expressed in its lowest terms, and also when the experimental data are sufficiently accurate. It can be shown that 1 prout is about 0.18163 Mev. We consider the evidence to be in favor of conclusions 4 and 5.

The magnetic moments of nuclei appear to be integral multiples of $\mu_N/4 \times 42$, where μ_N is the nuclear magneton. The experimental evidence for this is quite good.

The experimental value of $\beta \equiv M_p/m$ is compatible with the assumption that it is $(16/15) \times 42 \times 41$. If this is so, the mass of the electron is given by $m = 45/16$ prouts.

The value of the pure number

$$\gamma \equiv \frac{e^2}{GM_p^2}$$

is in the neighborhood of $e^{2\times 42}$. All this shows the great importance of the number 42.

These results naturally suggest the idea that perhaps all intrinsic observable nuclear quantities are rational multiples of a natural unit for that quantity. The term intrinsic nuclear quantity does not include such quantities as cross-sections where interaction with an external entity is involved.

The nature of the numerical relationships stated in this paper appears to require that *certain* integers like 42 and 7 enter *somehow* into the fundamental equations of physics. With one or two exceptions to be mentioned immediately, this is the first time in the history of physics that *particular* integers or rational numbers have entered into the fundamental equations of physics. One exception is the spin of the ultimate particles in Bohr units, which is $^1/_2$ for the proton, neutron and electron. The second exception, at least it can be counted as such, is the *three* dimensions of space and *one* of time. The integral and half-integral quantum numbers of quantum physics are not a case in point, because in that case the *whole set* of integers or half-integers enter into the fundamental theory and not merely certain ones. This appears to be an item of the utmost importance, because it may and probably will require a change in the *kind* of mathematical system used in formulating basic theory.

It seems improbable on the basis of these results that any theory of nuclear forces can be correct that does not at the same time account for the masses of the proton and neutron, and possibly all the other elementary particles, even mesotrons. None of the existing theories fulfil this criterion. The natural conclusion is that they are all inadequate. This does not necessarily mean that the existing theories are of no value whatsoever.

It seems altogether improbable that results such as those given in this paper could be obtained from a theory of the nucleus in which the constituent particles are held together by forces which are continuous functions of continuously varying space coördinates, *even with quantization*. It seems probable that the observed results can be obtained only from some theory in which all the quantities involved are rational or integral numbers. This seems to point to the necessity of "quantizing" the space and time coördinates. We use the word "quantization" here to mean any theory which by its very nature implies the existence of a minimum measurable distance and/or gives a discrete character to space-time. What the detailed nature of such "quantization" is remains to be seen.

Now this idea fits in quite well with the importance of the number 42. For the Riemann-Christoffel tensor of space-time has 21 *different* components. The Riemann-Christoffel tensor is one of the most fundamental tensors of a space. If we multiply 21 by 2 for some reason, possibly spin, we get the mysterious number 42. Since in the special theory of relativity the Riemann-Christoffel tensor is zero, this leads us to the idea that the explanation of nuclear forces requires a general theory of relativity, but a general theory in which space-time is "quantized." Now in Einstein's theory of general relativity it is not the Riemann-Christoffel tensor that is important, but the contracted Riemann-Christoffel tensor. This leads us to propose the idea that nuclear structure depends on the Riemann-Christoffel tensor of a quantized space-time. Thus a particle will be some kind of singularity in a quantized space-time and will have a finite size because of the quantization of space-time. Gravitation of course will also be determined by this Riemann-Christoffel tensor, and in fact in such a way that in the limit when the atomic structure of matter is neglected we will obtain the Einstein equation of gravitation.

It is hoped that our long article on this subject will appear very soon. It is our intention to continue our study of the experimental data of nuclear physics and to seek to develop a theoretical scheme along the lines indicated here.

[1] Birge, R. T., *Reviews of Modern Physics*, 13, 233–239 (1941).

[2] Pollard, E., and Davidson, W. L., *Applied Nuclear Physics*, John Wiley & Sons Inc., New York (1942).

[3] Ewald, Heinz, *Zeits. f. Naturforschung*, 1, 131–136 (1946).

ON BEHRBOHM AND PINL'S LINEARIZATION OF THE EQUATION OF TWO-DIMENSIONAL STEADY POLYTROPIC FLOW OF A COMPRESSIBLE FLUID

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The equation for the velocity potential of a compressible fluid in two-dimensional steady polytropic flow in Cartesian coördinates x, y is

$$[\lambda + (\mu - 1)\phi_x^2 + \mu\phi_y^2]\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + [\lambda + (\mu - 1)\phi_y^2 + \mu\phi_x^2]\phi_{yy} = 0, \tag{1}$$

where

$$\lambda \equiv c_0^2 + \frac{\kappa - 1}{2} q_0^2, \qquad \mu = \frac{1 - \kappa}{2}, \tag{2}$$

c_0 and q_0 being reference sound and flow speeds, and κ the polytropic exponent.[1] In a recent paper Behrbohm and Pinl,[2] observing that when $\lambda = 1$ and $\mu = 1$ the equation (1) reduces to that satisfied by minimal surfaces, have generalized the Minkowski support function[2] to achieve a "new" linearization of the equation (1). They have shown that if $\omega(\alpha, \beta, \gamma)$ is a function homogeneous of degree 1 satisfying the equation

$$\frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\beta^2} + \frac{1}{\lambda + (\mu - 1)\dfrac{\alpha^2 + \beta^2}{\gamma^2}} \frac{\partial^2\omega}{\partial\gamma^2} = 0, \tag{3}$$

then a solution of the equation (1) may be obtained in the parametric form

$$x = \frac{\partial\omega}{\partial\alpha}, \qquad y = \frac{\partial\omega}{\partial\beta}, \qquad \phi(x, y) = \frac{\partial\omega}{\partial\gamma}, \tag{4}$$

provided that after all differentiations have been carried out the parameters α, β, γ are connected by the relation

$$\mu(\alpha^2 + \beta^2) + \lambda\gamma^2 = (2\mu - 1)^{2\mu}\gamma^{2-2\mu}. \tag{5}$$

In the present note we proceed to show that the equation (3) is a simple alternative form of the equation obtained by the familiar linearization of the equation (1) by means of the Legendre transformation, that the equation (5) is superfluous, that hence we may set γ equal to 1 and obtain a physical interpretation for the variables α and β as Cartesian components of the velocity vector, and finally that the most natural method of solution

of the equation (3) leads directly to Tschaplygin's classical solutions of the equation (1).

As our point of departure we take the linearization of the equation (1) which is obtained by the Legendre transformation: If $f(u, v)$ satisfies the equation

$$[\lambda + (\mu - 1)v^2 + \mu u^2]f_{uu} + 2uvf_{uv} + [\lambda + (\mu - 1)u^2 + \mu v^2]f_{vv} = 0, \quad (6)$$

then a velocity potential $\phi(x, y)$ in Cartesian coördinates may be found from the parametric equations

$$x = f_u, \quad y = f_v, \quad \phi(x, y) = f - uf_u - vf_v. \quad (7)$$

The variables u and v are the Cartesian components of the corresponding velocity vector. Suppose now we introduce new variables α and β by the transformation

$$u = \frac{\alpha}{\gamma}, \quad v = \frac{\beta}{\gamma}, \quad (8)$$

where γ is a parameter. Suppose

$$\omega(\alpha, \beta, \gamma) \equiv \gamma f\left(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}\right). \quad (9)$$

Then ω is a homogeneous function of degree 1; the equations (7) become the equations (4); also

$$\omega_{\alpha\alpha} = \frac{1}{\gamma}f_{uu}, \quad \omega_{\alpha\beta} = \frac{1}{\gamma}f_{uv}, \quad \omega_{\beta\beta} = \frac{1}{\gamma}f_{vv}. \quad (10)$$

Substituting these expressions in the equation (6) we find that

$$\left[\lambda + (\mu - 1)\frac{\alpha^2 + \beta^2}{\gamma^2}\right][\omega_{\alpha\alpha} + \omega_{\beta\beta}] + \frac{1}{\gamma^2}[\alpha^2\omega_{\alpha\alpha} + 2\alpha\beta\omega_{\alpha\beta} + \beta^2\omega_{\beta\beta}] = 0. \quad (11)$$

But the homogeneity of $\omega(\alpha, \beta, \gamma)$ implies that

$$\alpha^2\omega_{\alpha\alpha} + 2\alpha\beta\omega_{\alpha\beta} + \beta^2\omega_{\beta\beta} = \gamma^2\omega_{\gamma\gamma}. \quad (12)$$

If we substitute this formula in the equation (11) we have at once Behrbohm and Pinl's equation (3). If we regard γ as a given function of α and β rather than as an independent variable, after a more extended computation we reach the same result. The interesting geometric and algebraic apparatus used by Pinl and Behrbohm in their derivation, as well as their restriction (5), is now seen to be superfluous. All our steps are reversible, so that the equations (3) and (6) are simple transforms of each other.

There are only two independent variables in our original problem, but

solutions of the system of equations (3), (12), will involve three. These variables may be left free, or they may be normalized in any convenient way. In the minimal surface problem it is customary to impose the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$ so as to make them direction cosines of the normal to the surface, but in the compressible flow problem it is more natural to put γ equal to 1 and thus by equations (8) to make α and β the Cartesian components of the velocity vector.

Behrbohm and Pinl's result is in fact not a new linearization of a nonlinear equation, but rather a transformation of an old one, an illustration of a general transformation principle in which the general linear differential equation of the second order in two independent variables may be reduced to a system of two linear partial differential equations of the second order in three independent variables. In the special problem we are considering here we merely make two observations. First, the degree of homogeneity of the function $\omega(\alpha, \beta, \gamma)$ is not essential. Suppose we have some solution of the equation (11) homogeneous of any degree, say k; if we divide it by γ^{k-1} it becomes homogeneous of degree 1, and remains a solution of the equation (11). The type of solution obtained is of course the same since in the end we put γ equal to 1. Second, the homogeneity of $\omega(\alpha, \beta, \gamma)$ has been used in the definition (9) and in the partial differential equation (12), but only the latter is essential, since if a function $\omega(\alpha, \beta, \gamma)$ satisfies both the equations (3) and (12), it will certainly satisfy equation (11), and hence $\omega(x, y, 1)$ will satisfy our original equation (7). The equation (12) is satisfied not only by functions given by the formula (9), but also by functions of form

$$\omega(\alpha, \beta, \gamma) = f(\alpha\gamma, \beta\gamma), \tag{13}$$

or by a linear combination of such functions with functions homogeneous of degree 1.

Behrbohm and Pinl's paper contains a second linearization of the equation (1) of a similar type, which I conjecture also to be a simple equivalent to the equation (6), but it is sufficiently complicated that it is unlikely to be useful in the computation of flows, so we shall not consider it here.

Although nothing can be learned from this "new" linearization which cannot be discovered by the ordinary hodograph method, we may regard the equation (3) as a convenient formal device to enable us to obtain solutions of the equation (6).

It is natural to seek solutions of the equation (3) in cylindrical coördinates:

$$\omega(\alpha, \beta, \gamma) = \pi_k(\rho, \gamma)[A_k \cos k\theta + B_k \sin k\theta], \tag{14}$$

where

$$\rho^2 \equiv \alpha^2 + \beta^2, \quad \tan\theta \equiv \frac{\beta}{\alpha}. \tag{15}$$

The equation for $\pi_k(\rho, \gamma)$ then is

$$\frac{\partial^2 \pi_k}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \pi_k}{\partial \rho} - \frac{k^2}{\rho^2}\pi_k + \frac{1}{\lambda + (\mu - 1)\dfrac{\rho^2}{\gamma^2}}\frac{\partial^2 \pi_k}{\partial \gamma^2} = 0. \qquad (16)$$

The most general function homogeneous of degree 1 in ρ and γ must be of the form

$$\pi_k(\rho, \gamma) = \gamma q^k f_k(q), \qquad (17)$$

where

$$q \equiv \frac{\rho}{\gamma}. \qquad (18)$$

Equation (16) now yields an ordinary differential equation for $f_k(q)$:

$$q[\lambda + \mu q^2]\frac{d^2 f_k}{dq^2} + [(2k + 1)(\lambda + \mu q^2) - q^2]\frac{df_k}{dq} + k(k - 1)q f_k = 0, \qquad (19)$$

which by the substitution

$$s \equiv -\frac{\mu}{\lambda}q^2 \qquad (20)$$

reduces to the hypergeometric equation

$$s(1 - s)\frac{d^2 f_k}{ds^2} + \left[k + 1 - \left(k + 1 - \frac{1}{2\mu}\right)s\right]\frac{df_k}{ds} - \frac{k(k - 1)}{4\mu}f_k = 0. \qquad (21)$$

Accordingly the most general solution homogeneous of degree 1 of the equation (3) which can be obtained in the form (14) is given by superposition of terms of the form

$$\omega_k(\rho, \gamma) = \gamma C_k\left(\frac{\rho}{\gamma}\right)^k F\left(\frac{k}{2} - \frac{1}{4\mu} + \frac{1}{2}l, \frac{k}{2} - \frac{1}{4\mu} - \frac{1}{2}l;\quad k + 1;\right.$$
$$\left. -\frac{\mu}{\lambda}\left(\frac{\rho}{\gamma}\right)^2\right) + \gamma D_k\left(\frac{\rho}{\gamma}\right)^{-k} F\left(-\frac{k}{2} - \frac{1}{4\mu} + \frac{1}{2}l, -\frac{k}{2} - \frac{1}{4\mu} - \frac{1}{2}l;\right.$$
$$\left. 1 - k;\quad -\frac{\mu}{\lambda}\left(\frac{\rho}{\gamma}\right)^2\right), \qquad (22)$$

where

$$l \equiv \sqrt{k^2\left(1 - \frac{1}{\mu}\right) + \frac{1}{4\mu^2}}, \qquad (23)$$

provided k is not an integer. If k is an integer the second solution will

involve a logarithmic term. After we place the inessential variable γ equal to 1, the solutions derived from this potential by the aid of formula (4) becomes identical with the classical solutions of Tschaplygin.[4]

In deducing the preceding solution we have used the homogeneous form (9) but we could equally well have used form (13); since $\tan\theta = (\beta\gamma)/(\alpha\gamma)$, instead of the formula (17) we should have had

$$\pi_k(\rho, \gamma) = (\rho\gamma)^k f_k(\rho\gamma), \tag{24}$$

and in the end our result would have been the same. In both cases, of course, the kth power of the variable is introduced as a matter of convenience only.

There are other coördinate systems in which the equation (3) will separate. One variable may be split off in any system in which azimuthal planes are one of the families of coördinate surfaces, but when the inessential variable γ is put equal to 1 the resulting solution will of necessity reduce to Tschaplygin's solution (22). I think it likely, but have so far been unable to prove, that in fact it is impossible to separate variables in the equation (3) in any system of coördinates in which azimuthal planes are not one of the coördinate families, and hence that Tschaplygin's solutions of the hodograph equation are the only ones obtainable by separation of the variables.

[1] The polytropic exponent is usually denoted by γ in English usage, but we follow the notation of reference (2) throughout.

[2] Behrbohm, H., and Pinl, M., "Neue Linearisierung der Grundgleichung der eben adiabatischen kompressiblen Potentialströmung," *Z.A.M.M.*, **21**, 193–203 (1941).

[3] Courant, R., and Hilbert, D., *Methoden der Mathematischen Physik*, Vol. 2, Berlin, 1937, pp. 44–46.

[4] Tschaplygin, S., *Gas Jets*, Moscow University Scientific Memoirs, 1902, pp. 1–121, translation published as *N.A.C.A. Technical Memorandum* No. 1063, Washington, 1944; see p. 19.

ERRATUM

In the article, "Reverse-Mutation and Adaptation in Leucineless *Neurospora*," these PROCEEDINGS, **32**, 163 (1946), lines 17–20 on page 165 should read: "In the 25 asci and among the 158 additional spores listed in table 2 there were no recombinations. Since these numbers test for 129 chances for recombination, of which none were fulfilled, the genes involved are probably alleles."